CONTINUOUS-TIME FRACTIONAL ORDER LINEAR SYSTEMS IDENTIFICATION USING CHEBYSHEV WAVELET

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Shuen Wang¹, Ying Wang⁵, Yinggan Tang⊗

¹College of Mechanical and Electrical Engineering, Hulunbuir University, Hailar District, Inner Mongolia, 021008, China
⊗Institute of Electrical Engineering, Yanshan University, Qinhuangdao, Hebei 066004, China

ABSTRACT

In this paper, the identification of continuous-time fractional order linear systems (FOLS) is investigated. In order to identify the differentiation or-ders as well as parameters and reduce the computation complexity, a novel identification method based on Chebyshev wavelet is proposed. Firstly, the Chebyshev wavelet operational matrices for fractional integration operator is derived. Then, the FOLS is converted to an algebraic equation by using the the Chebyshev wavelet operational matrices. Finally, the parameters and differentiation orders are estimated by minimizing the error between the output of real system and that of identified systems. Experimental results show the effectiveness of the proposed method.

Keywords: Identification, fractional order system, Chebyshev wavelet, Operational matrices, Optimization

Introduction

Building an effective and accurate mathematical model to characterize the system’s dynamic behavior is an important issue in many engineering fields, especially in the community of control. Traditionally, most of real systems were modeled by differential equations in the frame of integer order calculus (IOC). The integer order differential equations (IODE) are finite dimensional, in other words, they have local characteristic and short-term history memory.
Therefore, IODE cannot fully describe the adequate dynamics of complex systems in some times. As a branch of mathematics, fractional order calculus (FOC) is an extension of IOC to non-integer case. Different from IOC, FOC is non-local and it is able to emphasize mathe- matically the long-term history memory. Therefore, many real systems such as semi-infinite lossy transmission lines [1], diffusion of the heat through a semi-infinite solid [2, 3], viscoelastic systems [4] and dielectric polarization [5] are more suitable to be described by fractional order models (FOM) than integer ones.

In recent years, modeling real control systems by FOMs is more and more attractive to researchers. For examples, Podlubny built a FOM for a heat- ing furnace in [6], the lead acid battery was modeled by a FOM [7], and Wang et al. built a FOM for thermal process in the boiler main steam sys- tem [8]. At present, the practical and popular way of building a FOM for a control system is to system identification. The primary goal of fractional order system identification is to establish a FOM capable of reproducing sys- tem’s physical behaviour as faithfully as possible from a series of observations [9]. In the literature, many methods have been proposed for fractional order system identification. These methods can be roughly classified into two cat- egories, i.e., the time domain methods and the frequency domain methods. In time domain, the equation-error and output-error methods proposed in [10–13] are basic and typical methods. Simplified refined instrumental vari- able (SRIVC) method [14], subspace method [15] and set member method [16] have also been proposed for fractional order system identification. In frequency domain, the Levy’s identification method was extended by Val’erio et al. to identify fractional transfer function [17–19]. The commensurate and non-commensurate fractional transfer function were studied in [17] and [19], respectively. In [20], a robust estimation of FOM in frequency domain using set membership method was proposed. In [21], a subspace identifica- tion method in frequency domain was proposed for commensurate fractional order system identification. In [22], the identifiability of FOM in frequency domain was investigated.

Though great progress have been made in fractional order system iden- tification, several disadvantages are still associated with current researches. First, no matter for time or frequency methods, it is still difficult to iden- tify the differentiation (or integral) orders of fractional order systems [19], coupled with the parameters. Second, extensive computation burden is involved in the process of fractional order system identification. This is because the calculation of fractional derivative of input and output signals is more complex than integer derivative [23].

Operational matrices, which is constructed based on various orthogonal functions, have been widely adopted to deal with the problems of dynamic system such as the solution of systems, identification and optimal control, etc. [24–26]. The main characteristic of this technique is that it converts a differential equation into an algebraic one. Therefore, it not only simplifies the problem but also tremendously reduces the computational complexities. Recently, various operational matrices of fractional differentiation and integration operators have been developed. B-spline operational matrix [27], Bernstein operational matrix [28], Chebyshev operational matrix [29], block pulse operational matrix [30] and wavelets operational matrix [30–32] are just a few examples. However, these operational matrices were used to find numerical solution of various fractional differentiation equations.

Motivated by the above facts, a novel method based on Chebyshev wavelet operational matrix for FOLS identification is proposed in this paper. Compared to other orthogonal functions, Chebyshev wavelet has several attractive features. First, wavelet can provide accurate representation of many functions and operators. Second, it is local supported, and is very suitable to the analysis of system with abrupt variations. Furthermore, Chebyshev wavelet can be regarded as the hybridization of Haar wavelet and Chebyshev polynomial. First, the FOLS is converted to an algebraic equation via Chebyshev wavelet operational matrix. Then, the parameters and differentiation orders of the FOLS are simultaneously identified by minimizing the error between the output of the true system and that of identified system.

The rest of this paper is organized as follows. In section 2, some mathematical knowledge relative to fractional calculus are briefly introduced. The Chebyshev wavelets and their fractional integration operational matrix are given in section 3. The identification of FOS using Chebyshev wavelets are explained in section 4. The experimental results are given in section 5. Finally, conclusion remarks are given in section 6.

Mathematical background

Definitions of fractional derivatives and integrals

In this section, we give some necessary definitions which will be applied in this paper. Fractional calculus is a generalization of the integration and differentiation to non-integer order fundamental operator aDαt, where a and t are the limits and α (aR) is the order of the operation [23]. The operator is defined as
There are several definitions for fractional calculus. Among these definitions, the G-L definition and Riemann-Liouville (R-L) definition will be used in this paper. The G-L definition is given as

\[
a_D^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{[t-a]/h} \binom{q}{j} f(t-jh)
\]

where \( [\cdot] \) means the integer part, and

\[
\binom{q}{j} = \frac{(-1)^j \Gamma(\alpha + 1)}{\Gamma(j + 1) \Gamma(\alpha + j - 1)}.
\]

\( \Gamma() \) is the Euler’s Gamma function and \( h \) is the finite sampling interval. The R-L definition is given as [25],

\[
a_D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau
\]

where \( n-1 < \alpha < n, n \in N \). The fractional integration of R-L is given by

\[
(I_a^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau.
\]

Another useful tool for describing fractional order system is the Laplace transform. The Laplace transform of R-L fractional derivative is defined as [33]

\[
\mathcal{L}\{_0^a D_t^\alpha f(t)\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k [^0 D_t^{\alpha-k-1} f(t)]_{t=0}.
\]

Under zero initial condition, the Laplace transform of fractional derivative is simplified as

\[
\mathcal{L}\{_0^a D_t^\alpha f(t)\} = s^\alpha F(s).
\]

The Laplace transform of fractional integral under zero initial condition is given as

\[
\mathcal{L}\{I_0^\alpha f(t)\} = \frac{1}{s^\alpha} F(s).
\]

Chebyshev wavelet operational matrix of fractional integration

Chebyshev wavelet

\[
\psi_{a,b}(t) = |a|^{-1/2} \psi\left(\frac{t-b}{a}\right), a, b \in \mathbb{R}, a \neq 0,
\]

where \( \psi(t) \) is called the mother wavelet, \( a \) is the dilation parameter and \( b \) is the translation parameter. If the dilation and translation parameter are restricted to discrete values as \( a = a^k, b = na^kb_0 (a_0 > 1, b_0 > 0, n \) and \( k \)
are positive integers, one can obtain discrete wavelets as
\[
\psi_{k,n}(t) = |a_0|^{-k/2} \psi(a_0^k t - nb_0),
\]
(10)
Chebyshev wavelets are constructed from Chebyshev polynomials, they are defined on the internal [0,1) as
\[
\psi_{n,m}(t) = \begin{cases} 2^{k/2} U_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \leq t \leq \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise} \end{cases},
\]
where \(k\) is any positive integer, \(n = 1, 2, \ldots, 2^k - 1\), and \(U_m(t)\) is defined as
\[
U_m(t) = \begin{cases} 1/\sqrt{\pi}, & m = 0 \\ \sqrt{2/\pi} T_m(t), & m > 0, \end{cases}
\]
(12)
where \(m = 0, 1, \ldots, M - 1\), \(T_m(t)\) is the first kind of Chebyshev polynomials with the degree of \(m\), which are orthogonal with respect to the weight function \(w(t) = 1/(1 - t^2)\) on the interval \([-1,1]\).

Any square integrable function \(f(t)\) on the interval [0,1) be expanded onto Chebyshev wavelet series as
\[
f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(t),
\]
(14)
where \(c_{nm}\) is called wavelet coefficient and is given by
\[
c_{nm} = \langle f(t), \psi_{nm}(t) \rangle = \int_0^1 f(t) \psi_{nm}(t) \, dt.
\]
(15)
In practice, the series is truncated and one has an approximation of \(f(t)\) as
\[
f(t) \approx \sum_{n=1}^{2^{k-1} M-1} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(t) = C^T \Psi(t),
\]
(16)
where \(C\) and \(\Psi(t)\) are \(2^{k-1} M \times 1\) matrices given by
\[
C \triangleq \begin{bmatrix} c_{10}, c_{11}, \cdots, c_{1 M-1}, c_{20}, \cdots, c_{2 M-1}, \cdots, c_{2^{k-1} 0}, \cdots, c_{2^{k-1} M-1} \end{bmatrix}^T
\]
(17)
\[
\Psi(t) \triangleq \begin{bmatrix} \psi_{10}, \psi_{11}, \cdots, \psi_{1 M-1}, \psi_{20}, \cdots, \psi_{2 M-1}, \cdots, \psi_{2^{k-1} 0}, \cdots, \psi_{2^{k-1} M-1} \end{bmatrix}^T
\]
(18)
Chebyshev wavelet operational matrix of fractional integration
For a Chebyshev wavelet vector \(\Psi(t)\) in (18), if
\[
(\mathbf{I}_q^\alpha \Psi)(t) = P \Psi(t),
\]
(19)
then \(P\) is called the Chebyshev wavelet operational matrix of fractional integration, whose size is \(2^{k-1} M \times 2^{k-1} M\). In the following, we derive the Chebyshev wavelet operational matrix of fractional integration via the block functions. Since Chebyshev wavelets are
piecewise constant, they can be expended into \( m \)-term block pulse functions as

\[
\Psi_m(t) = \Phi_{m \times m} B_m(t),
\]  

where \( B_m(t) = [b_1(t), b_2(t), \ldots, b_m(t)]^T \) is block pulse function vector with

\[
b_i(t) = \begin{cases} 
1, & \frac{i}{m} \leq t < \frac{i+1}{m} \\
0, & \text{otherwise}
\end{cases}.
\]  

The block pulse operational matrix of fractional integration \( F^\alpha \) is [35]

\[
F^\alpha = \left( \frac{1}{m} \right)^\alpha \frac{1}{\Gamma(\alpha + 2)} \begin{pmatrix} f_1 & f_2 & f_3 & \cdots & f_M \\
0 & f_1 & f_2 & \cdots & f_{M-1} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & f_1 \end{pmatrix}_{m \times m}
\]  

where \( f_1 = 1, f_p = p^{\alpha+1}2(p1)^{\alpha+1} + (p2)^{\alpha+1} \). Take the R-L fractional integration in both sides of Eq.(20), one can obtain

\[
(I_0^\alpha \Psi_m)(t) = (I_0^\alpha \Phi_{m \times m} B_m)(t) = \Phi_{m \times m}(I_0^\alpha B_m)(t) = \Phi_{m \times m} F^\alpha B_m(t).
\]  

From Eq.(20), one has

\[
B_m(t) = \Phi^{-1}_{m \times m} \Psi_m(t).
\]  

Substitute Eq.(24) into Eq.(51) and we can get

\[
(I_0^\alpha \Psi_m)(t) = \Phi_{m \times m} F^\alpha \Phi^{-1}_{m \times m} \Psi_m(t),
\]  

therefore, we have the Chebyshev wavelet operational matrix of fractional integration as

\[
P^{\alpha}_{m \times m} = \Phi_{m \times m} F^\alpha \Phi^{-1}_{m \times m}.
\]  

The analytical expression of fractional integrable function \( f(t) \) can be expressed as

\[
(I^\alpha f)(t) = C^T P^\alpha \Psi_m(t).
\]  

By applying the Chebyshev operational matrix, one can convert the fractional integral of a function into an algebra operation, which can dramatically reduce the complexity of problems under consideration. The Chebyshev operational matrix of fractional derivative \( G \) can be obtained by inverting the matrix \( P \), i.e.,

\[
G^{\alpha} = P^{-\alpha}.
\]  

FOS identification using Chebyshev wavelet operational matrices

Consider a single input single output (SISO) linear time invariant (LTI) fractional order system described by the following differential equation,
The transfer function of the system (29) is given as

\[
H(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \cdots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_0 s^{\alpha_0}}.
\]  

(30)

where \(\alpha_i\) and \(\beta_j\) are arbitrary positive real numbers, \(u(t)\) and \(y(t)\) are the input and output of the system.

The goal of FOLS identification is to estimate the system parameters \(a_i, b_j\), and the differential orders \(\alpha_i\) and \(\beta_j\) according to the measured input and output data.

In this paper, the Chebyshev wavelet operational matrices of fractional integral is utilized for this purpose. To this end, both the numerator and denominator divide \(s^{\alpha_n}\), one can get

\[
H(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m - \alpha_n} + b_{m-1} s^{\beta_{m-1} - \alpha_n} + \cdots + b_0 s^{\beta_0 - \alpha_n}}{a_n + a_{n-1} s^{\alpha_{n-1} - \alpha_n} + \cdots + a_0 s^{\alpha_0 - \alpha_n}},
\]  

(31)

Eq.(31) can be expressed as

\[
\sum_{i=0}^{n} a_i [s^{\alpha_i - \alpha_n} Y(s)] = \sum_{j=0}^{m} b_j [s^{\beta_j - \alpha_n} U(s)].
\]  

(32)

Applying Eq.(27) to the system input and output, one can get

\[
(I_0^\alpha y)(t) \simeq C_Y^T P^\alpha \Psi_m(t),
\]  

(33)

And

\[
(I_0^\alpha u)(t) \simeq C_U^T P^\alpha \Psi_m(t).
\]  

(34)

Take the inverse Laplace transform of both sides of Eq.(32) and using Eqs.(33)-(34), one can get

\[
C_Y^T D \Psi_m(t) = C_U^T N \Psi_m(t),
\]  

(35)

Where

\[
D = a_0 P^{\alpha_n - \alpha_0} + a_1 P^{\alpha_n - \alpha_1} + \cdots + a_n I,
\]  

(36)

And

\[
N = b_0 P^{\beta_m - \alpha_n} + b_1 P^{\beta_{m-1} - \alpha_n} + \cdots + b_m I.
\]  

(37)

From Eq.(35), one has

\[
C_Y^T = C_U^T N D^{-1}.
\]  

(38)

Since

\[
y(t) = C_Y^T \Psi_m(t),
\]  

(39)

Therefore, we have
Eq. (40) provides an effective and simple way to calculate the output \( y(t) \) of fractional system (29). It is an algebraic operation instead of differential equation, which avoids complex calculation of fractional derivative of input and output signal. Furthermore, the matrix \( ND^{-1} \) contains the system parameters and the fractional differential orders. The above advantages enable us easily to construct an algorithm to identify the parameters and the fractional differential orders.

Let \( a^i, b^j, \alpha^i \) and \( \beta^j \) be the estimation of \( a_i, b_j, \alpha_i \) and \( \beta_j \). According to (40), the operational matrix representation of the output of the estimated system can be written as

\[
y(t) = C_U^T ND^{-1} \Psi_m(t),
\]

where \( N \) and \( D \) are the estimation of matrices \( N \) and \( D \). The optimal estimation of parameters can be obtained by minimizing the following objective function,

\[
\min_{(\hat{a}_i^r, \hat{b}_j^r, \hat{\alpha}_i^r, \hat{\beta}_j^r)} \frac{1}{L} \sum_{t=1}^{L} [y(t) - \hat{y}(t)]^2,
\]

where \( \Gamma \) is the admitted search range of system parameters, and \( L \) is the number of data point used for parameter estimation.

To solve the optimization problem (42), many conventional optimization techniques can be used. In this paper, fmincon function in MATLAB optimization toolbox is used. For the convenience of statement, let \( \theta = [a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m, \alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m] \) be the generalized parameter vector of system (16), which contains the parameters \( a_i, b_j \) and the fractional differential orders \( \alpha_i \) and \( \beta_j \). \( \theta^0 \) be the estimation of \( \theta \). The main steps for identifying the parameters and fractional differential orders of the fractional order system are summarized as follows:

**Step 1:** Preparing identification data. Exciting the original fractional system using an input signal \( u(t) \) and record the its corresponding output \( y(t) \).

**Step 2:** Let \( k = 0 \), give an initial guess of the estimated parameter vector \( \hat{\theta}^0 \), and calculate the output of estimated system according to Eq.(34).

**Step 3:** Performing an iterative process to get the next estimation \( \hat{\theta}^k \) using a certain optimization method.

**Step 4:** \( k = k + 1 \), and goto Step 3 until a termination criteria is satisfied.

**Simulation examples**

In this section, four identification examples are given to show the effectiveness of the proposed identification method. To quantify the estimation accuracy, the relative error (RE) of parameters identification and the mean square error (MSE) between the output of the true system are calculated. RE is defined as

\[
RE = \left\| \hat{\theta} - \theta \right\| / \left\| \theta \right\|.
\]

**Example 1**

Consider a FOLS as

\[
H(s) = \frac{Y(s)}{U(s)} = \frac{2.5}{1.2s^{2.5} + 1.5s^{1.3} + 0.7^3}.
\]

First, both the numerator and denominator divide \( s^{2.5} \), then, the transfer function of system (44) can be written as

\[
H(s) = \frac{2.5s^{-2.5}}{1.2 + 1.5s^{-1.2} + 0.7s^{-2.5}}.
\]

The coefficient of system (44) are \( a_0 = 0.7, a_1 = 1.5, a_2 = 1.2, b_0 = 2.5 \) and the integral order are \( a_0 = 2.5, a_1 = 1.2, a_2 = 0, b_0 = 2.5 \). A unit step signal is used as input to excite the system and record the input output data. In simulation, we select \( M = 4, k = 7 \). Therefore, the dimension of the Chebyshev wavelet operational matrix is \( m = 256 \).

And we can take 1000 collocation points on the interval \([0,10]\). Therefore, for arbitrary orders, the input
signal vector $U$ in Eq. (23) is $U = [1, 1, 1]^T$, whose length is equal to 256.

The identification of the parameters and orders is achieved by minimizing objective function (42). To this end, the MATLAB function fmincon with interior-point method is adopted. The identification results are listed in Table 1, which are compared with the Haar wavelets operational matrix method. The step response of the true system and the identification system are plotted in Fig.1, and the Bode diagrams are shown in Fig.2. It can be seen that the parameters of the identified system are in agreement with the true values, so the step responses are almost overlap. The frequency responses

| Table 1: Parameter identification results of Example 1 |
|----------------|----------------|----------------|----------------|
| Parameter      | True value     | Haar wavelets  | Chebyshev wavelets |
| $\alpha_0$     | 2.5            | 2.4991         | 2.4992          |
| $\alpha_1$     | 1.2            | 1.2011         | 1.2010          |
| $\alpha_2$     | 0              | 0              | 0               |
| $\beta_0$      | 2.5            | 2.4980         | 2.4981          |
| $a_0$          | 0.7            | 0.7054         | 0.7052          |
| $a_1$          | 1.5            | 1.5135         | 1.5128          |
| $a_2$          | 1.2            | 1.2169         | 1.2157          |
| $b_0$          | 2.5            | 2.5265         | 2.5249          |
| PE             | 0.0070         | 0.0066         |                 |
| MSE            | 5.4840e-05     | 5.0542e-05     |                 |

*Figure 1: Step responses of the true system and the identified system of Example 1*
of the two systems come to the same conclusion. Therefore, the identified models give satisfactory results.

The identified system is verified by using a sinusoidal \( u(t) = \sin(t) \) as the input signal to excite the system. The response of the two systems are shown in Fig. 3.

Figure 2: Bode diagram of the true system and the identified system of Example 1

Example 2

The second FOLS with the following transfer function is considered,

\[
H(s) = \frac{Y(s)}{U(s)} = \frac{2.3s^{1.3} + 3}{1.5s^{2.1} + 0.7s^{1.5} + 2.7s^{0.7} + 0.5}
\]

System (46) can be rewritten as

\[
H(s) = \frac{2.3s^{-0.8} + 3s^{-2.1}}{1.5 + 0.7s^{-0.6} + 2.7s^{-1.4} + 0.5s^{-2.1}}
\]

Similarly, the numerator and denominator of system (46) both divide \( s^{1.3} \), then system (46) can be rewritten as

The parameters of system (46) are \( a_0 = 0.5, a_1 = 2.7, a_2 = 0.7, a_3 = 2.3, b_0 = 3, b_1 = 2.3 \) and the orders are \( a_0 = 2.1, a_1 = 1.4, a_2 = 0.6, a_3 = 0, b_0 = 2.1, b_1 = 0.8 \).

As in Example 1, a unit step signal is used as input to excite the system. The identification process is the same as Example 1. The identification results are shown in

Figure 3: Sinusoidal responses of the true system and the identified system of Example 1
Table 2. The step response of the true system and that of the identified system is shown in Fig. 4 and the Bode diagram is shown in Fig. 5.

The identified system is verified by using an sinusoidal as input. The responses of the true system and the identified system for sinusoidal signal are shown in Fig. 6. It can be seen that the time and frequency responses of the two systems are very close. Therefore, the identification results are satisfactory.

Table 2: Parameter identification result of Example 2

<table>
<thead>
<tr>
<th>Parameter</th>
<th>True value</th>
<th>Haar wavelets</th>
<th>Chebyshev wavelets</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_0$</td>
<td>2.1</td>
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</tr>
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<td>$\alpha_3$</td>
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<td>0</td>
<td>0</td>
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<td>$\beta_0$</td>
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<td>2.0822</td>
<td>2.0859</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.8</td>
<td>0.7925</td>
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<tr>
<td>$a_0$</td>
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<td>3.0465</td>
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<td>$b_1$</td>
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<tr>
<td>PE</td>
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<tr>
<td>MSE</td>
<td></td>
<td>8.3603e-05</td>
<td>5.4363e-05</td>
</tr>
</tbody>
</table>

Figure 4: Step responses of the true model and the identified model of Example 2

Figure 5: Bode diagram of the true model and the identified model of Example 2
Example 3
Here, we consider an integer order system, which is a special case of fractional system. Its transfer function is given as

$$H(s) = \frac{1}{s^2 + 5s + 2}. \quad (48)$$

The numerator and denominator of the system both divide $s^2$, one gets

$$H(s) = \frac{s^{-2}}{1 + 3s^{-1} + 2s^{-2}} \quad (49)$$

The parameters of the system are $a_0 = 2$, $a_1 = 3$, $a_2 = 1$, $b_0 = 1$, and the orders are $\alpha_0 = 2$, $\alpha_1 = 1$, $\alpha_2 = 0$. The identification process is the same as the previous two examples. The estimated values are listed in Table 3. The step responses and Bode diagrams of the true system and the identified system are shown in Fig.7 and Fig.8, respectively. It can be seen that the time and frequency responses of the two systems are very close.

Example 4
Lastly, a heating furnace is considered, its transfer function is given as

$$G(s) = \frac{1}{14994s^{1.31} + 6009.5s^{0.97} + 1.69}. \quad (50)$$
Table 3: The parameter identification results of Example 3

<table>
<thead>
<tr>
<th>Parameter</th>
<th>True value</th>
<th>Identified value</th>
</tr>
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Figure 7: Step response of the true system and the identified system of Example 3

Figure 8: Bode diagram of the true system and the identified system of Example 3
The numerator and denominator both divide $s^{1.31}$, and one can get

$$\frac{Y(s)}{U(s)} = \frac{s^{-1.31}}{14994 + 6009.5s^{-0.34} + 1.69s^{-1.31}}. \tag{51}$$

The parameters of the system are $a_0 = 1.69$, $a_1 = 6009.5$, $a_2 = 14994$, $b_0 = 1$, the orders of the system are $a_0 = 1.31$, $a_1 = 0.34$, $a_2 = 0$ and $b_0 = 1.31$. The identification process is similar to the previous three examples. The identification results of parameters and the fractional differential orders are listed in Table 4. The step responses and Bode diagrams of the true system and the identified system are shown in Fig.10 and Fig.11, respectively.

<table>
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<th>Parameter</th>
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<th>Identified value</th>
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PE

MSE

Conclusion

In this paper, a novel method is proposed to identify FOLS based on Chebyshev wavelet operational matrix of the fractional integration. Several simulations are presented to demonstrate the efficiency of the methodology for fractional system identification. Compared with block pulse functions and Haar wavelets operational matrix method, Chebyshev wavelets operational matrix method has better accuracy with lower PE and MSE when applied to the identification of fractional system. The precision can be improved by increasing the dimension, but it comes at the cost of computational time.
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References


