DOUBLE TRANSFORM OF LAPLACE

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Regularity of the double transform of Laplace in the opened area of 0 is proved. The class of the transform of Laplace from the transform of Fourier is considered from the functions without a regularity in null.

Transform of Fourier, transform of Laplace, regularity of the double transform of Laplace, regularity of the transform of Laplace from the transform Fourier 1.

Introduction

We consider the regularity of the double transform of Laplace (the theorem 1, the remarks 1, 2). The theorem proved in this part have a generalmathematical character and are easily checked up. With help of the theorem 1 and the remark 2 it is simply to prove a some theorems related with the transform of Fourier and Laplace [1-6] (for instance, about the inverse operator of the transform of Laplace, using only positive values of the transform of Laplace on the [0,+∞) [5]). The theorems are not by the theme of the the article and require the separate study in connection with the theorem 1.

Some results in the direction were formulated in the works [4 ,5, 6,7].

The fact about double decomposition on the elementary fractions is considered conclusion. In opinion of author the fact underlines interest to the remark 2. By definition,

$$L_{\pm}Z(t)(\cdot)(x)=\int_0^\infty e^{\pm xt}Z(t)dt,x\in[0,\infty),$$

$$F_{\pm}u(t)(\cdot)(p)=\int_{-\infty}^{\infty}e^{\pm pit}u(t)dt,p\in(-\infty,\infty),L_+=L,$$

$$C^0u(t)(\cdot)(x)=\int_0^\infty \cos xtu(t)dt,S^0u(t)(\cdot)(x)=\int_0^\infty \sin xtu(t)dt,x\in(-\infty,\infty),$$

$$F^0_{\pm}u(t)(\cdot)(p)=\int_0^\infty e^{\pm pit}u(t)dt,p\in(-\infty,\infty).$$

2. The regularity of Laplace transform in $|z|<a>0$ In the section we use the Y1 condition.

The Y1 condition takes place for the $u(p)$ function, if the $u(p)$ function is regular for all p without only k points $z_1,...,z_k,z_j \notin (-\infty,\infty)\cup(-i\infty,i\infty),k=$ 0,1,..., $u(0)=0$, and

$\max[|u(p)|,|du(p)/dp|,|d^2u(p)/p^2|]|p^{2+\delta}|\to 0,|p|\to\infty.$

$\delta>0$, $\delta=const.$

We use the Ch1 condition too.

Ch1 condition.

The $u(p)$ function∩is regular in $K_{++}=\{p: Imp\ge 0\cap Rep\ge 0\}$ or in $K_{+-}=p:\ \{Imp\le 0\quad Rep\ge 0\}$.

Theorem 1

The

$$LF^0_+u(x)(\cdot)(z)=\int_0^\infty e^{-zt}dt\int_0^\infty e^{itx}u(x)dx=iLLu(x)(\cdot)(iz),\ LLu(x)(\cdot)(z),$$

functions are regular in the area $z:|z|<\varepsilon>0$ for some $\varepsilon>0$, if for the $u(p)$ function the Ch1, Y1 conditions take place.

Proof.

We can use the proposition 1 [5,6].

Proposition 1.

The

$LF_+^0u(x)(\cdot)(v)=iF_-^0Lu(x)(\cdot)(v),v\in[0,\infty)$, 0 0 0 0

$LC\,u(x)(\cdot)(v)=S\,Lu(x)(\cdot)(v),LS\,u(x)(\cdot)(v)=C\,Lu(x)(\cdot)(v),v\in[0,\infty)$

equalities take place, if for the $u(p)$ function the Y1 condition takes place

(The similar equality $LF^0_-u(x)(\cdot)(v)\ =\ -iF^0_+Lu(x)(\cdot)(v),v\ \in\ [0,\infty)$ takes place too). Proof.

We get the first formula after the change of order of integration in both parts of the first equality,(if $u(0)=0$, it is obviously wit help of the

$$|F^0_-u(x)(\cdot)(t)|\le|du(0)/dx/t^2|+|(1/t^2)F^0_-(d^2u(x)/dx^2)(\cdot)(t)|\le c_1/t^2,t\to\infty$$

expressions , ,$c_1=const.$, $c_1<\infty$, [8]).

With help of the proposition 1 we obtain, that the $F_{-}^{0}Lu(x)(\cdot)(p)=l_{-}(p)$ function is defined for all *Imp* < 0, and

$$\lim_{p\to iy, Im\,p<0} F_{-}^{0}Lu(x)(\cdot)(p)=F_{-}^{0}Lu(x)(\cdot)(iy), y\in(-\infty,\infty),$$

(it is obviously, if $u(0) = 0$; as in the proposition 1 we use the formula of integration on parts [8]).

Similar facts take place for a similar function $l_{+}=F_{+}^{0}Lu(x)(\cdot)(p)=$; the function is definite from other side of plane $Imp > 0$. We suppose $u(-p) = -u(p)$.

We can write

$$F_{-}^{0}Lu(x)(\cdot)(p)+F_{+}^{0}Lu(x)(\cdot)(p)=2C^{0}Lu(x)(\cdot)(p)=F(p), p=y, y\in[0,\infty),$$

if $u(-p) = -u(p)$, or

$$l_-(p) + l_+(p) = F(p),$$

where $F(p)$ are regular in $\{p : |Imp| < A\}\cup|Rep| < A\}$ for some $A > 0$, if the $u(p)$ function is regular as in the Y1 condition (the fact is well-known [2 , 5,6]).

To prove the fact for $p = y \in (-\infty,0]$ we can define a new functions $l_{-}^{to+}(p), l_{+}^{to-}(p)$:

$$l_-^{to+}(p) = l_-(p), Imp \le 0,$$

$$l_+^{to-} = l_+(p), Imp \ge 0,$$

where $l_{-}^{to+}(p)$ is an analytical continuation of the $l_-(p), Imp \le 0$, function $-$ $Im\,p\ge 0,$; l_{+}^{to-} from the lower part of plane to the overhead part of plane is an analytical continuation of the $l_+(p), Imp \ge 0$ function from from the overhead part of plane to the lower part of plane $Imp \le 0$ [2].

The $l_{-}^{to+}(p)+l_{+}(p)=F(p), Im\,p\ge 0$ equality repeats the main $l_-(p) + l_+(p) = F(p)$ equality, but in the $Imp \ge 0$ area; the $l_-(p)+l_{+}^{to-}(p)=F(p)$ equality repeats the main $l_-(p)+l_+(p) = F(p)$ equality, but in the $Imp \le 0$ area, where the $F(p)$ function is regular in $\{p : |Imp| < A\}\cup|Rep| < A\}$ [2]. We obtain, that

$$l_{-}^{to+}(p)+l_{+}(p)=l_{-}(p)+l_{+}^{to-}(p), p=y\in[0,\infty).$$

But the same equality takes place and for the $p = y \in (-\infty,0]$ (we use, that both functions $l_{-}^{to+}(p)+l_{+}(p)=F(p), l_{-}(p)+l_{+}^{to-}(p)=F(p)$ are equal to the regular $F(p)$ function in different parts of the plane [2]).

We get

$$l_{-}^{to+}(p)+l_{+}(p)=l_{-}(p)+l_{+}^{to-}(p), p=y\in(-\infty,\infty).$$

The $l_-(p), l_+(p)$ functions $l_{-}^{to+}(p), l_{+}(p), l_{-}(p), l_{+}^{to-}(p)$ are the transforms of Laplace in area of definition , and the functions are regular in area of definition [2] (from the proposition 1) with values on the boundary. The $l_{-}^{to+}(p), l_{+}^{to-}(p))$ functions are regular in the area of regularity of the sums from the the lemma 1.

Lemma 1.

1. The $LF_{+}^{0}u(x)(\cdot)(p)=iF_{-}^{0}Lu(x)(\cdot)(p)=il_{-}(p)=il_{-}^{to+}(p)$ function is regular for all $p : Imp \ge 0$ (we consider the branch of the function, passing through $p : p = iy, y = Imp > 0$ and $p : p = x, x = Rep > 0$, where the $F_{-}^{0}Lu(x)(\cdot)(p))$ values not defined), if for the $u(p)$ function the Y1,Ch1 conditions take place.
2. The $LF_{-}^{0}u(x)(\cdot)(p)=-iF_{+}^{0}Lu(x)(\cdot)(p)=-il_{+}(p)=l_{+}^{to-}(p)$ function is regular for all $p : Imp \le 0$ (we consider the branch of the function, passing through $p : p = iy, y = Imp < 0$ and $p : p = x, x = Rep > 0$, where the $F_{+}^{0}Lu(x)(\cdot)(p))$ values not defined), if for the $u(p)$ function the Y1,Ch1 conditions take place.

Proof.

If the $u(p)$ function is regular in$\cup$ $K_{++}\cup = \{p : Imp \ge 0 \cap Rep \ge 0\}$, after integration along the $L_+ = L_1 L_2 L_3$ line anticlockwise, $L_1 = [0,R]$, $L_2 = \{p : p = Re^{i\varphi}, 0 \le \varphi \le \pi/2\}$, $L_3 = [iR,0]$, we obtain, that

$$l_-^{to+}(p) = l_-(p) = F_-^0 Lu(x)(\cdot)(p) = \int_0^\infty (1/ip + x)u(x)dx =$$

$$= \int_0^\infty (1/ip + ix_1)u(ix_1)dix_1 = (1/i)LLu(ix_1)(\cdot)(p), p \in (0, +\infty),$$

as for $Rep \notin (-\infty,0)$ so as for all $Imp \geq 0$,. We use the Y1,Ch1 conditions (the $u(p)$ function is regular in $K_{++} = \{p : Imp \geq 0 \cap Rep \geq 0\}$; for

$Imp = 0$ we use the $u(0) = 0$ condition for proof of continuity on the $(-\infty,\infty)$ axis with help of the proposition 1 [8]).

We obtain, that the such sum is regular for all $Imp \geq 0, Reip \leq 0$, and

the

$$\int_0^\infty (1/ip + ix_1)u(ix_1)dix_1 = (1/i)LLu(x)(\cdot)(p), Re\,p \notin (-\infty, 0)$$

function is regular in the $Imp \geq 0, Reip \leq 0$ with the values on the boundary line $Imp = 0, Rep \leq 0$ (with help of the formula of integration on parts as in proposition 1 [2,8]).

For the $F_+^0 Lu(x)(\cdot)(p) = l_+(p) = l_+^{to-}(p), Im\,p < 0$ function we use

$l_+^{to-}(p) = \overline{l_-^{to+}(\overline{p})}$, as the branches of the $l_+(p) = l_-(p)$ functions [2] (with help

of the $\overline{F_+^0 Lu(x)(}\cdot)(p) = F^0 Lu(x)(\cdot)(p)$ formula on the $[0,+\infty)$ line) by the −

theorem of Riemann about the analytical continuation across the $(-\infty,\infty)$ line [2] $l_+(p) = l_+^{to-}(p) = overlinel_-^{to+}(p)(\overline{p})$, and the function is defined and regular in $Imp \leq 0$.

If the $u(p)$ function is regular∪in} $K \cup_{+-} = \{-p : Imp \leq 0 \cap Rep \geq 0\}$, after

integration{ along $^{i\varphi}$the,$-\pi/L+2 = \leq \varphi L1 \leq 0L2,_* L_3 * L = 3_*$ [lineiR,anticlockwise,0], we obtain,Lthat 1 = [0,R], $L_2* = p : p = Re$

$$F_+^0 Lu(x)(\cdot)(p) = l_+(p) = -\int_0^\infty (1/ip - x)u(x)dx =$$

$$- \int_0^{-\infty} (1/ip - ix_1)u(ix_1)dix_1 =$$

$$= \int_0^{+\infty} (1/ip + ix_2)u(-ix_2)dix_2 = LLu(-ix_1)(\cdot)(p), p \in (0, +\infty).$$

The further proof of lemma repeats proof of the first part.

With help of lemma 1 we get

$$|l_-^{to+}(p) + l_+(p)| \leq C = const., Im\,p \geq 0;\ l_-(p) + l_+^{to-}(p) \leq C = const., Im\,p \leq 0,$$

$C < \infty$. Both sums $l_-^{to+}(p) + l_+(p),\ l_-(p) + l_+^{to-}(p)$ are regular in in area of definition and continuous on the boundary [2,8].

We proved

$l^{to+}(p) + l_+(p) = C, C = const., C < \infty$ −

for all p [2], and $2C^0 Lu(x)(\cdot)(p) = F(p) = l^{to+}(p) + l_+(p) \equiv C_1 = const., C_1 <$

$$l_-^{to+}(p) = -l_+(p) \text{ for all p including } p \in (-\infty, \infty)$$

∞ [2,6] or.

We obtain, that the $F(p)$ function is regular [2] with the $2C\,^0 Lu(x)(\cdot)(p) = F(p), p \in (-\infty,\infty)$ values.

We can use, that the $LF_+^0 u(x)(\cdot)(p) = iF_-^0 Lu(x)(\cdot)(p)$ function is regular in $|Rep| < \varepsilon \cup Imp| < \varepsilon, \varepsilon > 0$, for some $\varepsilon > 0$(it is well-known fact [2 ,5, 6] in Y1 condition for the u(p) function). We proved, that

$$F_-^0 Lu(x)(\cdot)(p) = F(p) - F_+^0 Lu(x)(\cdot)(p)$$

is the analytical continuation from from one side of plane on other [2].

For the $u(-p) = u(p)$ we can use

$$F_-^0 Lu(x)(\cdot)(p) - F_+^0 Lu(x)(\cdot)(p) = 2iS^0 Lu(x)(\cdot)(p) = F(p), p = y, y \in [0, \infty),$$

further by analogy with the first part (with help of the lemma 1 and the $u(0) = 0$ condition).

The theorem 1 is proved.

From the theorem 1 we obtain the remark 1.

Remark 1.

The theorem 1 takes place for the $u(p) = v(p)+v(-p), u(p) = v(p)-v(-p)$ function, if all the essential points [2] of the $v(p)$ function are placed or in K_{++} (or all the essential points are in $K-+$), and for the $v(p)$ function the conditions of the theorem 1 take place.

From the remark 1 we get the first part of the remark 2.

The second part is easily proved by the methods of the work [1,6].

Remark 2.

1.

The theorem 1 takes place for the $u(p) = (v_1(p) - v_1(-p)) + (v_2(p) - v_2(-p))$ function, if the essential points [2] of the $v_i(p)$ function are placed or in K_{++} (or all the essential points are in $K-+$), $i = 1,2$, and for the $v_i(p)$ functions the conditions of the theorem 1 take place, i=1,2.

2.

The $u(-p) = -u(p)$, function can be presented in the $u(p) = (v_1(p) - v_1(-p))+(v_2(p)-v_2(-p))$ form, if for the $u(p)$ function the Y1 condition takes place;} the $LC^0u(x)(\cdot)(p)$ function is regular in the $\{p : |Imp| < A\} \cup |Rep| < A$ area.

3 .Conclusion

We will mark the fact about double decomposition on the elementary fractions:

$p[1/(p-1)-1/(p+1)] = 1/(p-1)+1/(p+1), p/(p-1)^2-p/(p^2-1) = 1/(p-1)^2+1/(p^2-1)$,

$p \neq 1,-1$. The fact in opinion of author underlines interest to the theorem 1 and to the consequences of the theorem 1.

Probably, the $|C^0S^0u(t)(\cdot)(x)| = |S^0C^0u(t)(\cdot)(x)|, x \in (0,\infty)$ equality ensues from theorem 1 and remark 2.

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